

Chapter 4

Vector differential equations

4.1 Problems VC-1

4.1.1 Topics of this homework:

Vector algebra and fields in \mathbb{R}^3 , gradient and scalar Laplacian operators, definitions of divergence and curl, Gauss's (divergence) and Stokes's (curl) laws, system classification (postulates).

4.1.2 Scalar fields and the ∇ operator

Problem # 1: Let $T(x, y) = x^2 + y$ be an analytic scalar temperature field in two dimensions (single-valued $\in \mathbb{R}^2$).

– 1.1: Find the gradient of $T(\mathbf{x})$ and make a sketch of T and the gradient.

Ans:

– 1.2: Compute $\nabla^2 T(\mathbf{x})$ to determine whether $T(\mathbf{x})$ satisfies Laplace's equation.

Ans:

– 1.3: Sketch the iso-temperature contours at $T = -10, 0, 10$ degrees.

Ans:

– 1.4: The heat flux¹ is defined as $\mathbf{J}(x, y) = -\kappa(x, y)\nabla T$, where $\kappa(x, y)$ is denotes the thermal conductivity at the point (x, y) . Given that $\kappa = 1$ everywhere (the medium is homogeneous), plot the vector $\mathbf{J}(x, y) = -\nabla T$ at $x = 2, y = 1$. Be clear about the origin, direction, and length of your result.

Ans:

– 1.5: Find the vector \perp to $\nabla T(x, y)$ —that is, tangent to the iso-temperature contours. Hint: Sketch it for one (x, y) point (e.g., 2, 1) and then generalize.

Ans:

– 1.6: The thermal resistance R_T is defined as the potential drop ΔT over the magnitude of the heat flux $|\mathbf{J}|$. At a single point the thermal resistance is

$$R_T(x, y) = -\nabla T/|\mathbf{J}|.$$

How is $R_T(x, y)$ related to the thermal conductivity $\kappa(x, y)$?

Ans:

Problem # 2: Acoustic wave equation

Note: In this problem, we will work in the frequency domain.

¹The heat flux is proportional to the change in temperature times the thermal conductivity κ of the medium.

– 2.1: *The basic equations of acoustics in one dimension are*

$$-\frac{\partial}{\partial x} \mathcal{P} = \rho_o s \mathcal{V} \quad \text{and} \quad -\frac{\partial}{\partial x} \mathcal{V} = \frac{s}{\eta_o P_o} \mathcal{P}.$$

Here $\mathcal{P}(x, \omega)$ is the pressure (in the frequency domain), $\mathcal{V}(x, \omega)$ is the volume velocity (the integral of the velocity over the wavefront with area A), $s = \sigma + \omega j$, $\rho_o = 1.2$ is the specific density of air, $\eta_o = 1.4$, and P_o is the atmospheric pressure (i.e., 10^5 Pa). Note that the pressure field \mathcal{P} is a scalar (pressure does not have direction), while the volume velocity field \mathcal{V} is a vector (velocity has direction).

We can generalize these equations to three dimensions using the ∇ operator

$$-\nabla \mathcal{P} = \rho_o s \mathcal{V} \quad \text{and} \quad -\nabla \cdot \mathcal{V} = \frac{s}{\eta_o P_o} \mathcal{P}.$$

– 2.2: *Starting from these two basic equations, derive the scalar wave equation in terms of the pressure \mathcal{P} ,*

$$\nabla^2 \mathcal{P} = \frac{s^2}{c_0^2} \mathcal{P},$$

where c_0 is a constant representing the speed of sound.

Ans:

– 2.3: *What is c_0 in terms of η_o , ρ_o , and P_o ?*

Ans:

– 2.4: *Rewrite the pressure wave equation in the time domain using the time derivative property of the Laplace transform [e.g., $dx/dt \leftrightarrow sX(s)$]. For your notation, define the time-domain signal using a lowercase letter, $p(x, y, z, t) \leftrightarrow \mathcal{P}$.*

Ans:

4.1.3 Vector fields and the ∇ operator**4.1.4 Vector algebra**

Problem # 3: Let $\mathbf{R}(x, y, z) \equiv x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}} + z(t)\hat{\mathbf{z}}$.

– 3.1: If a , b , and c are constants, what is $\mathbf{R}(x, y, z) \cdot \mathbf{R}(a, b, c)$?

Ans:

– 3.2: If a , b , and c are constants, what is $\frac{d}{dt} (\mathbf{R}(x, y, z) \cdot \mathbf{R}(a, b, c))$?

Ans:

Problem # 4: Find the divergence and curl of the following vector fields:

– 4.1: $\mathbf{v} = \hat{\mathbf{x}} + \hat{\mathbf{y}} + 2\hat{\mathbf{z}}$

Ans:

– 4.2: $\mathbf{v}(x, y, z) = x\hat{\mathbf{x}} + xy\hat{\mathbf{y}} + z^2\hat{\mathbf{z}}$

Ans:

– 4.3: $\mathbf{v}(x, y, z) = x\hat{\mathbf{x}} + xy\hat{\mathbf{y}} + \log(z)\hat{\mathbf{z}}$

Ans:

– 4.4: $\mathbf{v}(x, y, z) = \nabla(1/x + 1/y + 1/z)$

Ans:

4.1.5 Vector and scalar field identities

Problem # 5: Find the divergence and curl of the following vector fields:

– 5.1: $\mathbf{v} = \nabla\phi$, where $\phi(x, y) = xe^y$

Ans:

– 5.2: $\mathbf{v} = \nabla \times \mathbf{A}$, where $\mathbf{A} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$

Ans:

– 5.3: $\mathbf{v} = \nabla \times \mathbf{A}$, where $\mathbf{A} = y\hat{\mathbf{x}} + x^2\hat{\mathbf{y}} + z\hat{\mathbf{z}}$

Ans:

– 5.4: For any differentiable vector field \mathbf{V} , write two vector calculus identities that are equal to zero.

Ans:

– 5.5: What is the most general form a vector field may be expressed in, in terms of scalar Φ and vector \mathbf{A} potentials?

Ans:

Problem # 6: Perform the following calculations. If you can state the answer without doing the calculation, explain why.

– 6.1: Let $\mathbf{v} = \sin(x)\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$. Find $\nabla \cdot (\nabla \times \mathbf{v})$.

Ans:

– 6.2: Let $\mathbf{v} = \sin(x)\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$. Find $\nabla \times (\nabla \sqrt{\mathbf{v} \cdot \mathbf{v}})$

Ans:

– 6.3: Let $\mathbf{v}(x, y, z) = \nabla(x + y^2 + \sin(\log(z)))$. Find $\nabla \times \mathbf{v}(x, y, z)$.

Ans:

4.1.6 Integral theorems

Problem # 7: For each of the following problems, in a few words, identify either Gauss's or Stokes's law, define what it means, and explain the formula that follows the question.

– 7.1: What is the name of this formula?

$$\int_S \hat{\mathbf{n}} \cdot \mathbf{v} \, dA = \int_V \nabla \cdot \mathbf{v} \, dV.$$

Ans:

– 7.2: *What is the name of this formula?*

$$\int_S (\nabla \times \mathbf{V}) \cdot d\mathbf{S} = \oint_C \mathbf{V} \cdot d\mathbf{R}$$

Give one important application. **Ans:**

– 7.3: *Describe a key application of the vector identity*

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla(\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}.$$

Ans: